## REMARKS

This Amendment after Final Rejection is being filed in response to the Final Office Action mailed from the U.S. Patent and Trademark Office on March 1, 2004, in which claims 17-28, 49-60 and 63-82 were rejected. With this Amendment, independent claims 17, 49 and 61 are amended. In addition, Applicants note that there are two claims numbered 76 and no claim numbered 77, so Applicants have renumbered the second claim 76 as claim 77. As such, Applicants respectfully request reconsideration and allowance of pending claims 17-28 and 49-82.

Applicants respectfully believe the Office Action failed to include pending claims 61 and 62. For purposes of this response, Applicants have considered pending claims 61-62 to have the same rejections as the other pending claims, and have amended independent claim 61 to overcome the rejections of the Office Action.

The Office Action rejected claims 17-24, 26-28, 49-58 and 49-60 under 35 U.S.C. 102(e) as being anticipated by U.S. Patent No. 6,547,724 to Soble et al. ("the Soble et al. '724 patent"). Also, the Office Action rejected claims 63-82 under 35 U.S.C. 103(a) as being unpatentable over the Soble et al. '724 patent in view of U.S. Patent No. 5,391,144 to Sakurai et al. ("the Sakurai et al. '144 patent"). Also, the Office Action rejected claims 17-28, 49-60 and 63-82 under the judicially created doctrine of obviousness-type double patenting as being unpatentable over various combinations of Applicants' patents and copending patent applications.

### Anticipation Rejection Under 35 U.S.C. 102(e)

The Office Action rejected claims 17-24, 26-28, 49-58 and 49-60 under 35 U.S.C. 102(e) as being anticipated by the Soble et al. '724 patent. However, the Office Action later stated that the Soble et al. '724 patent does not teach the ultrasonic probe supporting a transverse ultrasonic vibration. The Office Action stated on page 9:

> **Soble et al. '724 do not expressly teach the probe supporting transverse ultrasonic vibration along at least a portion of the axial length of the probe** and wherein the vibration of the probe provides a plurality of anti-nodes along at least a portion of the axial length.

Applicants agree with the Examiner's statement that the Soble et al. '724 patent does not teach "the probe supporting transverse ultrasonic vibration." Instead, the Soble et al. '724 patent discloses a flexible sleeve slidingly transformable into a large suction sleeve for removing materials from body cavities, especially for removing stones from the urinary tract.

With this Amendment, Applicants have amended independent claims 17, 49 and 61 to claim an ultrasonic treatment apparatus comprising an ultrasonic probe wherein **at least a portion of the ultrasonic probe vibrates in a direction transverse to a longitudinal axis of the ultrasonic probe.** As the Office Action noted, the Soble et al. '724 patent does NOT disclose or suggest an ultrasonic treatment apparatus comprising an ultrasonic probe wherein **at least a portion of the ultrasonic probe vibrates in a direction transverse to a longitudinal axis of the ultrasonic probe.** Thus, the anticipation rejection regarding the Soble et al. '724 patent is overcome.

In addition, the Soble et al. '724 patent does not teach an aspiration channel recessed along the length of an outer surface of the ultrasonic probe as claimed in Applicants' claimed invention. The Office Action on page 8 asserts that Soble teaches an aspiration channel recessed along the length of an outer surface of the ultrasonic probe as follows:

> Sobel et al. '724 teach at least one aspiration channel recessed along the length of an outer surface of the ultrasonic probe, wherein aspiration occurs through the at least one aspiration channel along the length of the probe (see col. 6, lines 9-27).

Applicants respectfully disagree with the Office Action's above characterization of Soble et al. '724 patent. The Soble et al. '724 patent does <u>not</u> teach an aspiration channel recessed along the length of an outer surface of the ultrasonic probe.

Soble et al. '724 patent discloses a flexible sleeve for placing over a flexible endoscope with the sleeve containing lumens therein. As shown in FIGS. 3A, 3B and 3C the Soble lumens are located inside the sleeve 10. Further, the Office Action's reference to col. 6, lines 9-27 of Soble et al. '724 patent does not teach an aspiration channel recessed along the length of an outer surface of the ultrasonic probe as shown below:

> In another aspect of the invention, the **sleeve may contain multiple lumens defined by partition structures.** Apertures connected to these lumens may be part of the sleeve's distal opening, proximal opening, or its radial surface. As shown in **FIG. 3A, the sleeve 10 may contain, for example, three lumens defined by partition structure 6.** Lumen 1 is sized for insertion of the medical

> instrument (not shown). **One of the lumens can be used as an irrigation or ventilation channel** 3 connected to a source of pressurized fluid. Lumen 2 illustrates another working channel.
>
> The lumens can be substantially co-axial, as shown in FIG. 3B. All or one of the outer lumens may be used as the irrigation/ventilation channel 3 connected through an aperture (not shown) to a source of irrigation or ventilation. That aperture may be the port 20. The channel 3 can run along the length of the sleeve 10, which prevents the collapse of the cavity under treatment during suction by providing enough fluid flow to the cavity to counteract the vacuum caused by suction.

(Soble et al. '724 patent; col. 6, lines 9-27) (emphasis added). As the above passages show, Soble et al. '724 patent teaches lumens located inside a sleeve that can be used as an irrigation or ventilation channel.

The Soble et al. '724 patent does not teach **an aspiration channel recessed along the length of an outer surface of the ultrasonic probe**. Thus, the anticipation rejection regarding the Soble et al. '724 patent is overcome. Applicants respectfully request reconsideration and allowance of pending claims 17-28 and 49-82.

**Obviousness Rejection Under 35 U.S.C. 103(a)**

The Office Action rejected claims 63-82 under 35 U.S.C. 103(a) as being unpatentable over the Soble et al. '724 patent in view of the Sakurai et al. '144 patent. The Office Action stated on pages 8 and 9:

> In the same field of endeavor, Sakurai et al. '144 teach the probe supporting transverse ultrasonic vibration along at least a portion of the axial length of the probe and wherein the vibration of the probe provides a plurality of anti-nodes along at least a portion of the axial length (in Figure 13, see how the horn (element 63) transmits the vibrations to the ultrasonic probes (elements 61 or 62) with the transverse oscillation of the probe indicated in Figures 40 and 41 with indications of anti-node or loops wherein there is maximum oscillation along the length of the probes).
>
> It would have been obvious to one skilled in the art at the time that the invention was made to have modified Soble et al. '724 in view of Sakurai et al.' 144 and incorporated the specific teachings of Sakurai et al. '144 by using the specific probe as the probe of choice in lithotripsy by utilizing transverse vibration as this is such an ultrasonic device (see in col. 1, lines 12-16, describing use of device for breaking stones).

"Obviousness can only be established by combining or modifying the teachings of the prior art to produce the claimed invention where there is some teaching, suggestion, or motivation to do so found either explicitly or implicitly in the references themselves or in the knowledge generally available to one of ordinary skill in the art." M.P.E.P. 2143.01. "The test for an implicit showing is what the combined teachings, knowledge of one of ordinary skill in the art, and the nature of the problem to be solved as a whole would have suggested to those of ordinary skill in the art." *In re Kotzab,* 217 F.3d 1365, 1370, 55 U.S.P.Q.2d 1313, 1317 (Fed. Cir. 2000). See also *In re Fine*, 837 F.2d 1071, 5 U.S.P.Q.2d 1596 (Fed. Cir. 1988); *In re Jones*, 958 F.2d 347, 21 U.S.P.Q.2d 1941 (Fed. Cir. 1992); M.P.E.P. 2143.01.

Applicants respectfully disagree with the Examiner's statement that "Sakurai et al. '144 teach the probe supporting a transverse ultrasonic vibration." Examination of the Sakurai et al. '144 patent shows the Sakurai et al. device is a longitudinal device that does not operate in a transverse mode.

Differentiation between the Sakurai et al. longitudinal device and the Applicants' claimed transverse invention is evidenced by well known wavelength and stress equations, with further confirmation from the specification of the Sakurai et al. '144 patent.

**<u>Sakurai et al. Is A Longitudinal Device – Wavelength Equation Confirmation</u>**

Sakurai et al. '144 patent discloses an ultrasonic treatment apparatus that operates in longitudinal modes. Sakurai et al. discloses:

> The ultrasonic oscillator 2 can vibrate in various modes which are represented in the graph, i.e., the lower half of FIG. 40. As may be understood from the graph, the oscillator 2 has indefinite number of vibration modes. **Among these vibration modes are: the fundamental mode; the second harmonic mode in which ultrasonic waves have half the length of those in the fundamental mode; the third harmonic mode in which ultrasonic waves have a third of the length of those in the fundamental mode; and the fourth harmonic mode in which ultrasonic waves have a quarter of the length of those in the fundamental mode.** (Sakurai et al. '144 patent; Col. 23, Line 64 - Col. 24, Line 7)(emphasis added).

The above statement in Sakurai et al. conforms with the equation for the wavelength of **longitudinal waves** produced in a rod:[1]

$$\lambda_n = \frac{2\pi c}{\omega_n}$$ where $c$ is the speed of sound and $\omega_n$ is the angular frequency

Substituting the expression for the angular frequency $\omega_n$ in terms of the linear frequency, $\nu_n$,

$$\omega_n = 2\pi \nu_n$$

gives

$$\lambda_n = \frac{c}{\nu_n}$$

and finally substituting the expression for normal mode frequencies,

$$\nu_n = \frac{nc}{2L}$$

yields

$$\lambda_n = \frac{2L}{n}$$ where L is the length of the rod.

Using this equation for **longitudinal waves**, the fundamental harmonic wavelength is $\lambda_1$=2L, the second harmonic wavelength is $\lambda_2$=L and the third harmonic wavelength is $\lambda_3$=2L/3. Now the ratios of the second harmonic mode to the fundamental mode and the third harmonic mode to the fundamental mode can be calculated as follows:

**$\lambda_2/\lambda_1$ = L/2L = 0.50** (Ratio of Second Harmonic Mode to Fundamental Harmonic Mode for a **Longitudinal Wave**)

**$\lambda_3/\lambda_1$ = (2L/3)/2L = 0.33** (Ratio of Third Harmonic Mode to Fundamental Harmonic Mode for a **Longitudinal Wave**)

---

[1] Morse, P.M. and Ingard, K.U., *Theoretical Acoustics*, Princeton University Press, Princeton, NJ, pp. 116-120, 1968 (see attached Exhibit A).

FIG. 40 of Sakurai et al. is reproduced below with annotations A, B, $\lambda_1/2$, $\lambda_2/2$ and $\lambda_3/2$.

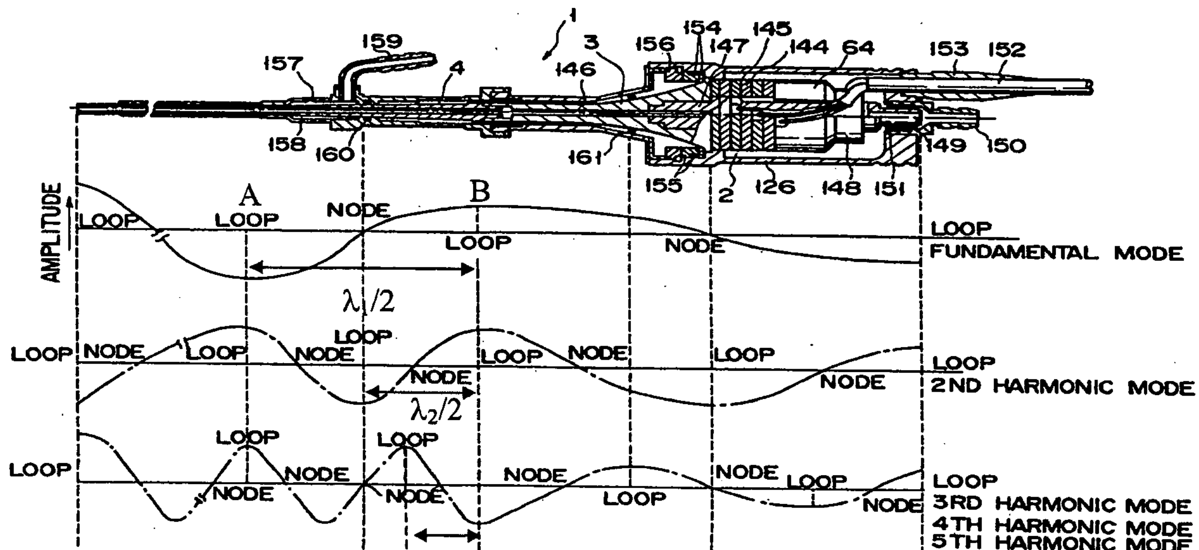


FIG. 40

FIG. 40 of Sakurai et al. '144 patent shows plots of amplitude of axial displacement of a longitudinal wave versus position for a longitudinal wave traveling along the device. For a given length L, for example between points A and B, along the transmission member, the Sakurai et al. device follows the equation for the wavelength of longitudinal waves discussed above. In viewing FIG. 40 of Sakurai et al. between points A and B, $\lambda_1=2L$, $\lambda_2=L$ and $\lambda_3=2L/3$. Thus, the ratios for the longitudinal waves $\lambda_2/\lambda_1 = 0.5$ and $\lambda_3/\lambda_1 = 0.33$ hold true for the Sakurai et al. device which operates in longitudinal modes. Further, FIG. 40 lists the fundamental mode, 2nd harmonic mode and 3rd harmonic mode. The fundamental, 2nd and 3rd modes are harmonically related which means the wavelengths are integer multiples of each other, further evidence that Sakurai et al. device operates in longitudinal modes.

The plot of the amplitude of axial displacement of a longitudinal wave versus position for a longitudinal wave traveling along the device in FIG. 40 of Sakurai et al. '144 patent showing the loops and nodes of the longitudinal motion of the Sakurai et al. device DOES NOT depict the device physically moving up and down. This plot is an abstract representation of the maximum deviation from equilibrium of a point at the given position along the longitudinal axis of the device. The plot merely shows an amplitude of axial displacement of a longitudinal wave versus position for a longitudinal wave traveling along the device to the tip.

In FIGS. 40-42 of the Sakurai et al. '144 patent, the progression of the wavelengths from fundamental mode to 2nd harmonic mode to 3rd harmonic mode shows the integer multiple relationship between the wavelengths described in the preceding analysis, and not the anharmonic relationship between the transverse modes which will be discussed below.

**Transverse Wavelength Equations**

The equation for **transverse wavelength** can be obtained by examining the solution for the mode shapes of a transverse wave:[2]

$$Y = a[\cosh(2\pi\mu\, x) - \cos(2\pi\mu\, x)] + b[\sinh(2\pi\mu\, x) - \sin(2\pi\mu\, x)]$$

where $Y$ is the amplitude of the wave at position $x$ on the rod, $a$ and $b$ are constants depending upon the specific nature of the problem and $2\pi\mu$ is the wavenumber, $k$, of the transverse wave. Using the relationship between the wavelength and the wavenumber

$$k = 2\pi\mu = \frac{2\pi}{\lambda}$$

gives

$$\lambda = \frac{1}{\mu}$$

Substituting the expression for $\mu_n{}^2$ gives the expression for the wavelength of the $n^{th}$ transverse mode:

$$\lambda_n = \frac{2L}{\beta_n} \qquad \text{with } \beta_n = \begin{cases} 0.597 & n=1 \\ 1.494 & n=2 \\ n-\tfrac{1}{2} & n>2 \end{cases}$$

where $L$ is the length of the rod.

Using this equation for **transverse waves**, the fundamental mode wavelength is $\lambda_1$=3.35L, the second mode wavelength is $\lambda_2$=1.34L and the third mode wavelength is $\lambda_3$=0.8L.

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[2] Morse, P.M. and Ingard, K.U., *Theoretical Acoustics*, Princeton University Press, Princeton, NJ, pp. 181-182, 1968 (see attached Exhibit A).

Now the ratios of the second mode to the fundamental mode and the third mode to the fundamental mode can be calculated as follows:

| | |
|---|---|
| $\boldsymbol{\lambda_2/\lambda_1 = 1.34L/3.35L = 0.40}$ | (Ratio of Second Mode to Fundamental Mode for a **Transverse Wave**) |
| $\boldsymbol{\lambda_3/\lambda_1 = 0.8L/3.35L = 0.24}$ | (Ratio of Third Mode to Fundamental Mode for a **Transverse Wave**) |

Comparing the ratios for longitudinal and transverse waves of the second mode and the third mode to the fundamental mode shows the inequality of the ratios for the varying waves. Therefore, further examination of the inequality of the ratios for the longitudinal waves and the transverse waves clearly shows that the Sakurai et al. device is a longitudinal device:

| **(Longitudinal Waves) (Sakurai et al. '144 Patent)** | | **(Transverse Waves)** |
|---|---|---|
| $\boldsymbol{\lambda_2/\lambda_1 = 0.50}$ | $\neq$ | $\boldsymbol{\lambda_2/\lambda_1 = 0.40}$ |
| $\boldsymbol{\lambda_3/\lambda_1 = 0.33}$ | $\neq$ | $\boldsymbol{\lambda_3/\lambda_1 = 0.24}$ |

As the above analysis shows, examination of equations for longitudinal waves, the Sakurai et al. '144 specification and FIGS. 40-42 all confirm that Sakurai et al. is a longitudinal device.

**Sakurai et al. Is A Longitudinal Device – Stress Equation Confirmation**

Further evidencing that Sakurai et al. is a longitudinal device is the specification of the Sakurai et al. '144 patent and well known stress equations. The Sakurai et al. '144 patent specification discusses a statement applicable to longitudinal waves, but not applicable to transverse waves:

> As is evident from FIG. 42, the junction between the horn 3 and the member 4, the first junction 171, and the second junction 177 are located at the loops of the ultrasonic vibration of the vibration-transmitting member 4. As is generally known in the art, less stress is applied to those portion of any member which are located at the loops of ultrasonic vibration, than to those portions which are located at the nodes of the vibration. Hence, a relatively small stress acts on the junction between the horn 3 and the member 4, the first junction 171, and the second junction 177--all being mechanically weak. (Sakurai et al. '144; Col. 25, Line 63 - Col. 26, Line 5).

The above statement is only true for longitudinal waves, and not transverse waves. Therefore, for the longitudinal Sakurai et al. device, there is little stress at the members located at the loops.

The well known equation for stress in a rod deformed **transversely** is:[3]

$$\sigma = \frac{Mc}{I} = Ec\frac{d^2v}{dx^2}$$

where $\sigma$ is the stress, $M$ is the bending moment, $c$ is the thickness of the rod, $I$ is the moment of inertia of the rod, E is the Young's modulus of the rod, $v$ is the transverse displacement of the rod, and $x$ is the coordinate along the axis of the rod.

For a rod vibrating in a normal **transverse** mode, the spatial distribution of the transverse displacement amplitude can be approximated by a sinusoidal wave pattern as follows:

$v = A\sin(kx)$

substituting this into the stress equation yields:

$\sigma = -Eck^2 A\sin(kx)$

Therefore, for a transverse mode device, it is clear that the maximum values for the stress are coincident with the points of maximum displacement ($kx = ((2n+1)\pi)/2$), which are defined as the loops and the minimum values for the stress are coincident with the points of minimum displacement, which are defined as the nodes. Conversely, the longitudinal mode Sakurai et al. device incurs maximum stress at the nodes. Therefore, the stress equations confirm Sakurai et al. is not a transverse device.

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[3] Craig, R.R. Jr., *Mechanics of Materials*, John Wiley and Sons, New York, NY, p. 268, 1996 (see attached Exhibit B).

**Sakurai et al. Is A Longitudinal Device And Energy Transfer Occurs At Tip**

The Sakurai et al. device operates in a longitudinal mode. The longitudinal mode of operation can only efficiently deliver energy to a surrounding medium at its tip. The longitudinal device operates by pushing material in an oscillatory fashion along the longitudinal axis of the rod only. This means that material along the side of a longitudinally oscillating rod is affected far less than material at the tip. Along the side, the rod is moving parallel to the surface of the adjacent material and little mechanical energy other than frictional heat is transferred from the rod into the surrounding material. At the tip, the rod is moving perpendicular to the surface of the adjacent material and much more mechanical energy can be imparted. Regarding FIG. 40 of the Sakurai et al. '144 patent (reproduced above), the only effective loop for transferring mechanical energy along a longitudinal device would be the one at the tip. Any other loop along the device is rendered ineffective because it can only vibrate parallel to the surface of a material with which it is in contact.

Sakurai et al. '144 discloses an ultrasonic treatment apparatus operating in a longitudinal mode that emulsifies tissue at the tip of the Sakurai et al. device. Sakurai et al. discloses:

> The **tips** of probes of different types, which can be attached to the hand piece, must be vibrated at different amplitudes to emulsify different types of tissues within body cavities, each type with the highest possible efficiency. (Sakurai et al. '144; Col. 12, Lines 24-28) (Emphasis added)
>
> If the **tip** of the probe 61 is in contact with a living tissue within a body cavity, it **emulsifies the tissue**." (Sakurai et al. '144; Col. 13, Lines 16-17)
>
> The **tip** of the member 4 is vibrated, and can therefore **cut or emulsify an affected tissue** or can break stones in a body cavity." (Sakurai et al. '144; Col. 23, Lines 61-63) (Emphasis Added)

Sakurai et al. discloses a device that operates in a longitudinal mode. Sakurai et al. does NOT disclose or suggest an ultrasonic probe wherein **at least a portion of the ultrasonic probe vibrates in a direction transverse to a longitudinal axis of the ultrasonic probe.**. The Sakurai et al. '144 patent does not cure or offer a suggestion on how to overcome the deficiencies of the Soble et al. '724 patent. Applicants respectfully request withdrawal of the obvious rejections to claims 63-82 and allowance of pending claims 17-28 and 49-82.

## Double Patenting Rejection

The Office Action rejected claims 17-28 and 49-82 under the judicially created doctrine of obviousness-type double patenting as being unpatentable based on claims of five issued patents and seven pending patent applications: U.S. Patent No. 6,524,251; U.S. Patent No. 6,660,013; U.S. Patent No. 6,652,547; U.S. Patent No. 6,679,873; U.S. Patent No. 6,695,781; copending Application No. 10/071,953; copending Application No. 10/268,487; copending Application No. 10/268,843; copending Application No. 10/371,781; copending Application No. 10/373,134; copending Application No. 10/396,914 and copending Application No. 10/462,182 (please note that Application No. 10/462,182 is U.S. Patent No. 6,679,873 listed above).

Applicants believe the present application claims separate and distinct subject matter than the five patents and seven pending patent applications. Thus, Applicants respectfully request withdrawal of all double patenting rejections because the present application claims subject matter that is patentably distinct from the claimed subject matter in the five patents and seven applications.

### The Present Application Claims Subject Matter That Is Patentably Distinct

The MPEP definition of Obviousness-Type Double Patenting states:

> Obviousness-type double patenting requires rejection of an application claim when the claimed subject matter is **not patentably distinct** from the subject matter claimed in a commonly owned patent when the issuance of a second patent would provide unjustified extension of the term of the right to exclude granted by a patent. *See Eli Lilly & Co. v. Barr Labs., Inc.*, 251 F.3d 955, 58 USPQ2d 1865 (Fed. Cir. 2001); *Ex parte Davis*, 56 USPQ2d 1434, 1435-36 (Bd. Pat. App. & Inter. 2000).

MPEP 804(II)(B)(1) (emphasis in original).

Applicants respectfully state that the present application claims subject matter that is patentably distinct from the claimed subject matter in the five patents and seven patent applications. Thus, the present application is not claiming common subject matter with the five patents and seven patent applications.

The present application is entitled "Method and Apparatus for Ultrasonic Medical Treatment, in Particular, for Debulking the Prostrate." The present application relates to an apparatus and a method for using an ultrasonic medical device operating in a transverse mode to debulk the prostrate.

Independent claims 17, 49, 61 and 73 of the present application all include the following limitation:

> an at least one aspiration channel recessed along the length of an outer surface of the ultrasonic probe, wherein aspiration occurs through the at least one aspiration channel along the length of the ultrasonic probe

The aspiration channel recessed along the length of an outer surface of the ultrasonic probe assists in debulking the prostrate. The claims in the five patents and seven applications do not contain such a limitation. The limitation is not an obvious variation of the patented claims. Thus, the present application claims subject matter that is patentably distinct from the claimed subject matter in the five patents and seven applications. Thus, Applicants respectfully request withdrawal of all double patenting rejections and reconsideration and allowance of claims 17-28 and 49-82.

Regarding Application No. 10/396,914, the Office Action's reference to the Soble et al. '724 patent does not make obvious the aspiration channel recessed along the length of an outer surface of the ultrasonic probe. As discussed above, the Soble et al. '724 patent does not teach an aspiration channel recessed along the length of an outer surface of the ultrasonic probe as claimed in Applicants' claimed invention. The Office Action on page 6 asserts that Soble teaches an aspiration channel recessed along the length of an outer surface of the ultrasonic probe as follows:

> In the same field of endeavor, Sobel et al. '724 teach at least one aspiration channel recessed along the length of an outer surface of the ultrasonic probe, wherein aspiration occurs through the at least one aspiration channel along the length of the probe (see col. 6, lines 9-27).

Applicants respectfully disagree with the Office Action's above characterization of Soble et al. '724 patent. Soble discloses a flexible sleeve for placing over a flexible endoscope with the sleeve containing lumens therein. As shown in FIGS. 3A, 3B and 3C the Soble lumens are located inside the sleeve 10. Further, the Office Action's reference to col. 6, lines 9-27 of Soble

et al. '724 patent does not teach an aspiration channel recessed along the length of an outer surface of the ultrasonic probe as shown below:

> In another aspect of the invention, the **sleeve may contain multiple lumens defined by partition structures.** Apertures connected to these lumens may be part of the sleeve's distal opening, proximal opening, or its radial surface. As shown in **FIG. 3A, the sleeve 10 may contain, for example, three lumens defined by partition structure 6.** Lumen 1 is sized for insertion of the medical instrument (not shown). **One of the lumens can be used as an irrigation or ventilation channel** 3 connected to a source of pressurized fluid. Lumen 2 illustrates another working channel.
>
> The lumens can be substantially co-axial, as shown in FIG. 3B. All or one of the outer lumens may be used as the irrigation/ventilation channel 3 connected through an aperture (not shown) to a source of irrigation or ventilation. That aperture may be the port 20. The channel 3 can run along the length of the sleeve 10, which prevents the collapse of the cavity under treatment during suction by providing enough fluid flow to the cavity to counteract the vacuum caused by suction.

(Soble et al. '724 patent; col. 6, lines 9-27) (emphasis added). As the above passages show, Soble et al. '724 patent teaches lumens located inside a sleeve that can be used as an irrigation or ventilation channel.

Soble et al. '724 patent does not teach **an aspiration channel recessed along the length of an outer surface of the ultrasonic probe**. Thus, it would <u>not</u> have been obvious to one skilled in the art to modify the claims of the patent application based on Soble et al. '724 patent to claim an aspiration channel recessed along the length of an outer surface of the ultrasonic probe.

Thus, the present application claims subject matter that <u>is patentably distinct</u> from the claimed subject matter in the five patents and seven applications. Thus, Applicants respectfully request withdrawal of all double patenting rejections and reconsideration and allowance of claims 17-28 and 49-82.

## <u>Conclusion</u>

In summary, the cited prior references, alone or in combination, do not anticipate, suggest, or make obvious Applicants' claimed invention in pending claims 17-28 and 49-82 and the rejections in the Office Action should accordingly be withdrawn. Reconsideration and allowance of pending claims 17-28 and 49-82 is respectfully requested.

Applicants have made an earnest effort to respond to all issues raised in the Office Action of March 1, 2004, and to place all claims presented in condition for allowance. No amendment made was for the purpose of narrowing the scope of any claim, unless Applicants have argued herein that such amendment was made to distinguish over a particular reference or combination of references.

Applicant submits that all claims are allowable as written and respectfully request early favorable action by the Examiner. If the Examiner believes that a telephone conversation with Applicants' attorney would expedite prosecution of this application, the Examiner is cordially invited to call the undersigned attorney of record.

Respectfully submitted,

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# THEORETICAL ACOUSTICS

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## 4.3 SIMPLE-HARMONIC OSCILLATIONS

It has been seen in the last section that imposing boundary conditions limits the sorts of motion that a string can have, and that if the boundary conditions correspond to the fixing of both ends of the string to rigid supports, the motion is limited to *periodic* motion. The latter result is an unusual one, for we found in the last chapter that even as simple a system as a pair of coupled oscillators does not, in general, move with periodic motion. It is not unusual for a system to oscillate with simple-harmonic motion (which is a special type of periodic motion) when it is started off properly (we shall see that practically every vibrating system can do this); what is unusual in the string between rigid supports is that *every* motion is periodic, no matter how it is started.

Our problem in this section is to find the possible simple-harmonic oscillations of the string (the normal modes of vibration) and to see what the relation is between the frequencies of these vibrations that makes the resulting combined motion always periodic. The problem of determining the normal modes of vibration of a system is not just an academic exercise. For systems more complicated than that of the string between rigid supports, we have no method of graphical analysis similar to that of the last section, and the only feasible method of discussing the motion is to "take it apart" into its constituent simple-harmonic components. There is also a physiological reason for studying the problem, for the ear itself analyzes a sound into its simple-harmonic parts (if there are any). We distinguish between a note from a violin and a note from a bell, for instance, because of this analysis. If the frequencies present in a sound are all integral multiples of a fundamental frequency, as they are in a violin, the sound seems more musical than when the frequencies are not so simply related, as in the note from a bell.

### Traveling and standing waves.

We start our discussion with the wave equation (4.1.2), which, we showed, determines the motion of a perfectly flexible string as long as it is not displaced too far from equilibrium (as long as $|\partial y/\partial x| \ll 1$).

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad c^2 = \frac{T}{\epsilon} \tag{4.3.1}$$

The wave equation corresponds to a number of statements concerning the motion of a string. We saw in the last section that it implies that the wave motion travels with its shape unchanged, at a velocity $c$, independent of this shape. Since the derivative $\partial^2 y/\partial x^2$ is proportional to the curvature of the shape of the string at a given instant, Eq. (4.3.1) states that the *acceleration*

of any portion of the string is *directly proportional to the curvature* of that portion. If the curvature is downward, the acceleration is downward, and vice versa; and the greater the curvature, the faster the velocity changes.

If the string is infinite in extent, it can carry waves which travel exclusively in one direction. In that case, as was pointed out at the beginning of this chapter, if the time dependence of the wave is to be sinusoidal, its space dependence must also be sinusoidal. All simple-harmonic waves traveling in the positive $x$ direction must have the form

$$y(x,t) = A \cos\left[\frac{\omega}{c}(x - ct) - \Phi\right]$$

or

$$y(x,t) = C \exp\left[\frac{i\omega}{c}(x - ct)\right] \tag{4.3.2}$$

if $C = Ae^{-i\Phi}$ and if physical meaning is attached only to the real part of the second expression. For a simple-harmonic wave in the negative $x$ direction we substitute $-(x + ct)$ for $(x - ct)$ in these expressions. Incidentally, the reason we have chosen the time factor to be $e^{-i\omega t}$ rather than $e^{j\omega t}$ is that then the sign of the $x$ part of the exponent, $e^{\pm i\omega x/c}$, indicates the direction of the wave.

For the wave of Eq. (4.3.2), the energy and momentum densities are [Eqs. (4.1.9) and (4.1.12)]

Kinetic energy density $= U = \frac{1}{2}\epsilon\omega^2 A^2 \sin^2\left[\frac{\omega}{c}(x - ct) - \Phi\right]$

Potential energy density $= V = \frac{1}{2}T\left(\frac{\omega}{c}\right)^2 A^2 \sin^2\left[\frac{\omega}{c}(x - ct) - \Phi\right]$

Total energy density $= W_{tt} = \epsilon\omega^2 A^2 \sin^2\left[\frac{\omega}{c}(x - ct) - \Phi\right] = H$ (4.3.3)

Energy flux $= W_{tx} = cH$

Longitudinal momentum density $= W_{xt} = \dfrac{H}{c}$

Longitudinal stress $= W_{xx} = \dfrac{H}{c^2}$

The energy density is greatest where the string's slope and transverse velocity are greatest, each packet of energy spaced a half wavelength from its neighbor, each traveling with a velocity $c$. Consequently, the energy flux $W_{tx}$ is equal to $c$ times the energy density $W_{tt}$. The wavelength of these waves is, of

course, the distance between one wave peak and the next, a distance such that an increase of $x$ by $\lambda$ will increase $(\omega/c)(x - ct)$ by $2\pi$, so that $(\omega/c)\lambda = 2\pi$, or $\lambda = 2\pi c/\omega$.

We could reach the same conclusions by asking what sort of shape the string will have when it vibrates with simple-harmonic motion, i.e., when its time dependence is through the factor $e^{-i\omega t}$. Setting $y(x,t) = Y(x)e^{-i\omega t}$ into the equation of motion (4.1.3) or (4.1.8), we obtain a familiar equation for $Y(x)$.

$$\frac{d^2Y}{dx^2} + \left(\frac{\omega}{c}\right)^2 Y = 0 \qquad c^2 = \frac{T}{\epsilon} \tag{4.3.4}$$

which is identical with Eq. (1.2.1). This is the equation for simple-harmonic dependence on $x$, with "angular frequency" $k = \omega/c$ and "period" $\lambda = 2\pi/k = 2\pi c/\omega$. The quantity $k$ is called the *wavenumber* of the wave; its dimensions are inverse length. The quantity $\lambda$ is the wavelength of the wave, the distance from crest to crest of a sinusoidal wave traveling in one direction.

The general solution of Eq. (4.3.4) can be written

$$Y(x) = C_+e^{i\omega x/c} + C_-e^{-i\omega x/c}$$

so that

$$\begin{aligned} y(x,t) &= C_+e^{i\omega(x-ct)/c} + C_-e^{-i\omega(x+ct)/c} \\ &= A_+\cos\left[\frac{\omega}{c}(x-ct) - \Phi_+\right] + A_-\cos\left[\frac{\omega}{c}(x+ct) + \Phi_-\right] \end{aligned} \tag{4.3.5}$$

representing two waves, of the same frequency and wavelength, traveling in opposite directions along the string. Since the wave equation is linear, neither wave has any effect on the other.

This mutual independence of the waves extends to expressions for their energy-momentum-stress terms, such as the total energy,

$$H = U + V = \omega^2(\theta_+^2 + 2\theta_+\theta_- + \theta_-^2)$$

where

$$\theta_+ = A_+\sin\left[\frac{\omega}{c}(x-ct) - \Phi_+\right] \qquad \theta_- = A_-\sin\left[\frac{\omega}{c}(x+ct) + \Phi_-\right]$$

The mean value of the square terms is $\langle\theta_+^2\rangle = \frac{1}{2}A_+^2$ and $\langle\theta_-^2\rangle = \frac{1}{2}A_-^2$, neither of which is zero. But the cross terms can be written

$$\theta_+\theta_- = \tfrac{1}{2}A_+A_-\left[\cos(2\omega t + \Phi_+ + \Phi_-) - \cos\left(2\frac{\omega}{c}x - \Phi_+ + \Phi_-\right)\right]$$

When averaged over space and time, the average is zero. The energy flux has no cross term;

$$W_{tx} = -T\frac{\partial y}{\partial t}\frac{\partial y}{\partial x} = \epsilon c\omega^2(\theta_+^2 - \theta_-^2)$$

so that even the instantaneous values of the flux are simply the differences between the two individual fluxes. Thus the average values of the stress-energy tensor are

$$H = \langle W_{tt} \rangle = \tfrac{1}{2}\epsilon\omega^2(A_+^2 + A_-^2) = \frac{1}{c^2}\langle W_{xx} \rangle$$

$$\Upsilon = \langle W_{tx} \rangle = \tfrac{1}{2}\epsilon\omega^2 c(A_+^2 - A_-^2) = \frac{1}{c}\langle W_{xt} \rangle \tag{4.3.6}$$

The energy and stress terms are the sum of the terms arising from each wave. The energy and momentum fluxes are the difference of the terms, since the two waves are flowing in opposite directions.

If the amplitudes of the two simple-harmonic waves are equal, there is no net flow of energy or momentum, and the combination is called a *standing wave*.

$$y(x,t) = C_+ e^{i\omega(x-ct)/c} + C_- e^{-i\omega(x+ct)/c}$$

$$= 2A \cos\left(\frac{\omega}{c}x + \tfrac{1}{2}\Phi_+ - \tfrac{1}{2}\Phi_-\right) \exp\left(-i\omega t + \tfrac{1}{2}\Phi_+ + \tfrac{1}{2}\Phi_-\right) \qquad \text{(real part)}$$

$$= 2A \cos\left(\frac{\omega}{c}x + \tfrac{1}{2}\Phi_+ - \tfrac{1}{2}\Phi_-\right) \cos\left(\omega t - \tfrac{1}{2}\Phi_+ - \tfrac{1}{2}\Phi_-\right) \tag{4.3.7}$$

where $C_+ = Ae^{i\Phi_+}$ and $C_- = Ae^{i\Phi_-}$ have the same amplitude but different phases. In this case the shape of the wave does not move along the string; it simply oscillates in amplitude with simple-harmonic motion. At points where $\cos\left[(\omega/c)x + \tfrac{1}{2}\Phi_+ - \tfrac{1}{2}\Phi_-\right] = 0$, the two traveling waves always cancel each other and the string never moves. These points are called the *nodal points* of the wave motion. In the case that we are considering, where the density and tension are uniform, the nodal points are equally spaced along the string a distance $c/2\nu$ apart, two for each wavelength. Halfway between each pair of nodal points is the part of the string having the largest amplitude of motion, where the two traveling waves always add their effects. This portion of the wave is called a *loop*, or *antinode*.

We should ask how a standing wave gets established and is maintained, for if there is no motion at each node, there can be no flow of energy from one loop to its neighbors. The answer is that a standing wave is a steady-state situation. During the transient state, when energy is being distributed along the string, the nodes are not perfect (that is, $y$ is not exactly zero there) and energy does pass from one loop to the next. Also, even for the steady-state situation, the nodes are only perfect when there is no friction. With zero friction, once a loop has acquired its energy, it can oscillate forever. If friction is present, the "nodes" are simply places of minimal (but not zero) amplitude of vibration; some energy flows from loop to loop.

## Normal modes

So far, we have neglected boundary conditions. If we require that $y = 0$ when $x = 0$, the general form of (4.3.5) can no longer be used; the number of possible harmonic motions is limited. The expression for $y$ that must be used is the standing-wave form (4.3.7) with the angles $\Phi$ so chosen that *a nodal point coincides with the point of support* $x = 0$:

$$y = A \sin\left(\frac{2\pi\nu}{c}x\right)\cos(2\pi\nu t - \Phi) \tag{4.3.8}$$

This agrees with the discussion in the previous section. For the simple boundary condition that we have used, the reflected wave has the same amplitude as the incident wave; and when the incident one is sinusoidal, the result is a set of standing waves. Any frequency is allowed, however.

When the second boundary condition $y = 0$ at $x = l$ is added, the number of possible simple-harmonic motions is still more severely limited. For now, of all the possible standing waves indicated in (4.3.8), *only those which have a nodal point at* $x = l$ *can be used*. Since the distance between nodal points depends on the frequency, the string fixed at both ends cannot vibrate with simple-harmonic motion of any frequency; only a discrete set of frequencies is allowed, the set that makes $\sin[(2\pi\nu/c)l]$ zero. The distance between nodal points must be $l$, or it must be $l/2$ or $l/3$, etc. The allowed frequencies are therefore $c/2l$, $2c/2l$, $3c/2l$, etc., and the different allowed simple-harmonic motions are all given by the expression

$$y = A_n \sin\left(\frac{\pi n x}{l}\right)\cos\left(\frac{\pi n c}{l}t - \Phi_n\right) \qquad n = 1, 2, 3, 4, \ldots$$
$$\nu_n = \frac{nc}{2l} = \frac{n}{2l}\sqrt{\frac{T}{\epsilon}} \tag{4.3.9}$$

The lowest allowed frequency $\nu_1 = c/2l$ is called the *fundamental frequency* of vibration of the string. It is the frequency of the general periodic motion of the string, as we showed in the last section. The higher frequencies are called *overtones*, the first overtone being $\nu_2$, the second $\nu_3$, and so on.

The equation for the allowed frequencies given in Eq. (4.3.9) expresses an extremely important property of the uniform flexible string stretched between rigid supports. It states that the frequencies of all the overtones of such a string are *integral multiples of the fundamental frequency*. Overtones bearing this simple relation to the fundamental are called *harmonics*, the fundamental frequency being called the first harmonic, the first overtone (twice the fundamental) being the second harmonic, and so on.

Very few vibrating systems have harmonic overtones, but these few are the bases of nearly all musical instruments. For when the overtones are harmonic, the sound seems particularly satisfying, or musical, to the ear.

The general solution of this is

$$\begin{aligned} y &= C_1 e^{2\pi\mu x} + C_2 e^{-2\pi\mu x} + C_3 e^{2\pi i\mu x} + C_4 e^{-2\pi i\mu x} \\ &= a\cosh(2\pi\mu x) + b\sinh(2\pi\mu x) + c\cos(2\pi\mu x) + d\sin(2\pi\mu x) \end{aligned} \tag{5.1.11}$$

where $\cosh u = \cos(iu)$ and $\sinh u = -i\sin(iu)$. See Eq. (1.2.10) and Tables I and II.

This general solution satisfies Eq. (5.1.10) for any value of the frequency $\nu$. It is, of course, the boundary conditions that pick out the set of allowed frequencies.

### Bar clamped at one end

For example, if we have a bar of length $l$ clamped at one end $x = 0$, the boundary conditions at this end are that *both* $y$ and its slope $\partial y/\partial x$ must be zero at $x = 0$. The particular combination of the general solution (5.1.11) that satisfies these two conditions is the one with $c = -a$ and $d = -b$.

$$Y = a[\cosh(2\pi\mu x) - \cos(2\pi\mu x)] + b[\sinh(2\pi\mu x) - \sin(2\pi\mu x)] \tag{5.1.12}$$

If the other end is free, $y$ and its slope will not be zero, but the bending moment $M = QS\kappa^2(d^2Y/dx^2)$ and the shearing force $F = -QS\kappa^2(d^3Y/dx^3)$ must both be zero, since there is no bar beyond $x = l$ to cause a moment or a shearing stress. We see that *two* conditions must be specified for each end instead of just one, as in the string. This is due to the fact that the equation for $Y$ is a fourth-order differential equation, and its solution involves four arbitrary constants whose relations must be fixed, instead of two for the string. It corresponds to the physical fact that whereas the only internal stress in the string is tension, the bar has two, bending moment and shearing force, each depending in a different way on the deformation of the bar.

The two boundary conditions at $x = l$ can be rewritten as $\dfrac{1}{4\pi^2\mu^2}\dfrac{d^2Y}{dx^2} = 0$ and $\dfrac{1}{8\pi^3\mu^3}\dfrac{d^3Y}{dx^3} = 0$ at $x = l$. Substituting expression (5.1.12) in these, we obtain two equations that fix the relationship between $a$ and $b$ and between $\mu$ and $l$:

$$a[\cosh(2\pi\mu l) + \cos(2\pi\mu l)] + b[\sinh(2\pi\mu l) + \sin(2\pi\mu l)] = 0$$

$$a[\sinh(2\pi\mu l) - \sin(2\pi\mu l)] + b[\cosh(2\pi\mu l) + \cos(2\pi\mu l)] = 0$$

or

$$b = a\,\frac{\sin(2\pi\mu l) - \sinh(2\pi\mu l)}{\cos(2\pi\mu l) + \cosh(2\pi\mu l)} = -a\,\frac{\cos(2\pi\mu l) + \cosh(2\pi\mu l)}{\sin(2\pi\mu l) + \sinh(2\pi\mu l)} \tag{5.1.13}$$

By dividing out $a$ and multiplying across, we obtain an equation for $\mu$:

$$[\cosh(2\pi\mu l) + \cos(2\pi\mu l)]^2 = \sinh^2(2\pi\mu l) - \sin^2(2\pi\mu l)$$

Utilizing some trigonometric relationships, this last equation can be reduced to two simpler forms:

$$\cosh(2\pi\mu l)\cos(2\pi\mu l) = -1 \qquad \text{or} \qquad \coth^2(\pi\mu l) = \tan^2(\pi\mu l) \quad (5.1.14)$$

where $\coth z = \cosh z/\sinh z$.

### The allowed frequencies

We shall label the solutions of this equation in order of increasing value. They are $2\pi\mu_1 l = 1.8751$, $2\pi\mu_2 l = 4.6941$, $2\pi\mu_3 l = 7.8548$, etc. To simplify the notation, we let $1/\pi$ times the numbers given above have the labels $\beta_n$, so that

$$\mu_n = \frac{\beta_n}{2l} \quad (5.1.15)$$

where $\beta_1 = 0.597$, $\beta_2 = 1.494$, $\beta_3 = 2.500$, etc. It turns out that $\beta_n$ is practically equal to $n - \frac{1}{2}$ when $n$ is larger than 2.

By fixing $\mu$, we fix the allowed values of the frequency. Using Eq. (5.1.10), we have

$$\nu_n = \frac{\gamma^2\mu_n^2}{2\pi} = \frac{\pi}{2l^2}\sqrt{\frac{Q\kappa^2}{\rho}}\,\beta_n^2 \quad (5.1.16)$$

or

$$\nu_1 = \frac{0.55966}{l^2}\sqrt{\frac{Q\kappa^2}{\rho}} \qquad \begin{aligned} \nu_2 &= 6.267\nu_1 \\ \nu_3 &= 17.548\nu_1 \\ \nu_4 &= 34.387\nu_1 \\ &\cdots\cdots \end{aligned}$$

Notice that the allowed frequencies depend on the inverse *square* of the length of the bar, whereas the allowed frequencies of the string depend on the inverse first power.

Equation (5.1.16) shows how far from harmonics are the overtones for a vibrating bar. The first overtone has a higher frequency than the sixth harmonic of a string of equal fundamental. If the bar were struck so that its motion contained a number of overtones with appreciable amplitude, it would give out a shrill and nonmusical sound. But since these high-frequency overtones are damped out rapidly, the harsh initial sound will quickly change to a pure tone, almost entirely due to the fundamental. A tuning fork can be considered to be two vibrating bars, both clamped at their lower ends. The fork exhibits the preceding behavior, the initial metallic "ping" rapidly dying out and leaving an almost pure tone.

### The characteristic functions

The characteristic function corresponding to the allowed frequency $\nu_n$ is given by the equation

$$\psi_n = a_n\left(\cosh\frac{\pi\beta_n x}{l} - \cos\frac{\pi\beta_n x}{l}\right) + b_n\left(\sinh\frac{\pi\beta_n x}{l} - \sin\frac{\pi\beta_n x}{l}\right) \quad (5.1.17)$$

where

$$-b_n = a_n \frac{\cosh(\pi\beta_n) + \cos(\pi\beta_n)}{\sinh(\pi\beta_n) + \sin(\pi\beta_n)} = a_n \frac{\sinh(\pi\beta_n) - \sin(\pi\beta_n)}{\cosh(\pi\beta_n) + \cos(\pi\beta_n)}$$

We shall choose the value of $a_n$ so that $\int_0^l \psi_n^2\, dx = l/2$, by analogy with the sine functions for the string. The resulting values for $a_n$ and $b_n$ are $a_1 = 0.707$, $b_1 = -0.518$, $a_2 = 0.707$, $b_2 = -0.721$, $a_3 = 0.707$, $b_3 = -0.707$, etc. For $n$ larger than 2, both $a_n$ and $b_n$ are practically equal to $1/\sqrt{2}$. Some of the properties of these functions that will be of use are

$$\int_0^l \psi_m(x)\psi_n(x)\, dx = \begin{cases} 0 & m \neq n \\ \dfrac{l}{2} & m = n \end{cases} \qquad \psi_n(l) = (-1)^{n-1}\sqrt{2}$$

$$\left(\frac{d\psi_1}{dx}\right)_{x=l} = 1.040\,\frac{\pi\beta_1}{l} \qquad \left(\frac{d\psi_2}{dx}\right)_{x=l} = -1.440\,\frac{\pi\beta_2}{l}$$

$$\left(\frac{d\psi_n}{dx}\right)_{x=l} \simeq (-1)^{n-1}\sqrt{2}\,\frac{\pi\beta_n}{l} \quad \text{and} \quad \beta_n \simeq n - \tfrac{1}{2} \qquad n > 2 \tag{5.1.18}$$

$$\psi_n \simeq \frac{1}{\sqrt{2}}\left[e^{-\pi\beta_n x/l} + (-1)^{n-1}e^{\pi\beta_n(x-l)/l}\right] + \sin\left(\frac{\pi\beta_n x}{l} - \frac{\pi}{4}\right) \qquad n > 2$$

The shapes of the first five characteristic functions are shown in Fig. 5.4. Note that for the higher overtones most of the length of the bar has the sinusoidal shape of the corresponding normal mode of the string, with the nodes displaced toward the free end. In terms of the approximate form given above for $\psi_n$, the sine function is symmetrical about the center of the bar; the first exponential alters the sinusoidal shape near $x = 0$ enough to make $\psi_n$ have zero value and slope at this point; and the second exponential adds enough near $x = l$ to make the second and third derivatives vanish. Note also that the number of nodal points in $\psi_n$ is equal to $n - 1$, as it is for the string.

In accordance with the earlier discussion of series of characteristic functions, we can now show that a bar started with the initial conditions, at $t = 0$, of $y = y_0(x)$ and $\partial y/\partial t = v_0(x)$ will have a subsequent shape given by the series

$$y = \sum_{n=1}^{\infty} \psi_n(x)[B_n \cos(2\pi\nu_n t) + C_n \sin(2\pi\nu_n t)] \tag{5.1.19}$$

where

$$B_n = \frac{2}{l}\int_0^l y_0(x)\psi_n(x)\, dx$$

$$C_n = \frac{1}{\pi\nu_n l}\int_0^l v_0(x)\psi_n(x)\, dx$$

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## 6.3 FLEXURAL STRESS IN LINEARLY ELASTIC BEAMS

In the previous section, assumptions were made about the geometry of deformation of slender beams, and an expression for the resulting extensional strain $\epsilon_x$ was derived, Eq. 6.3. The corresponding normal stress in beams, $\sigma_x$, is often called the *flexural stress*. To obtain an expression for the flexural stress, we need to consider the material behavior, that is, the stress-strain-temperature behavior of the material. To simplify our initial study of stresses in beams, let us assume that the material is linearly elastic and isotropic, and that the temperature remains constant. Then, the following two assumptions permit us to determine the flexural stress $\sigma_x$:

1. The material obeys Hooke's law, Eq. 2.32a, with $\Delta T = 0$.
2. The transverse normal stresses, $\sigma_y$ and $\sigma_z$, may be neglected in comparison with the primary normal stress, $\sigma_x$.

By combining these two assumptions, we find that the uniaxial stress-strain equation

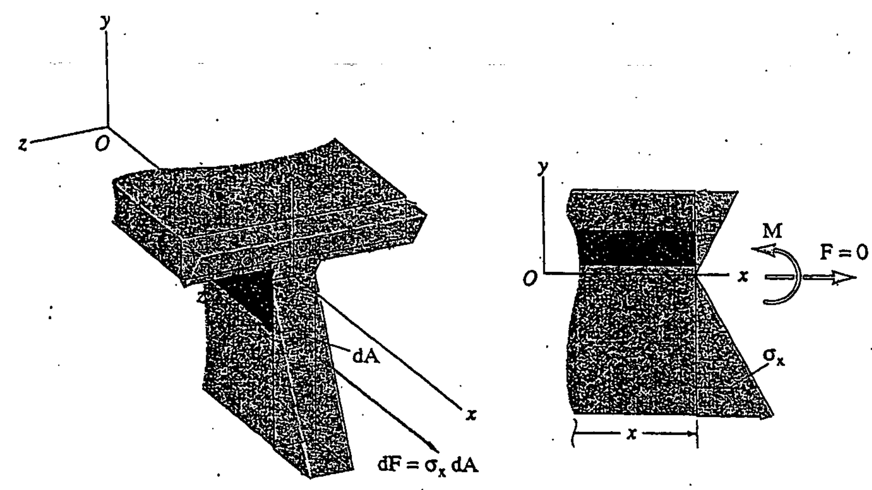
$$\sigma_x = E\epsilon_x \tag{6.6}$$

applies to bending of linearly elastic beams. When Eqs. 6.3 and 6.6 are combined, we obtain the following expression

$$\sigma_x = \frac{-Ey}{\rho} \tag{6.7}$$

If $E$ = const, or if $E = E(x)$, the normal stress on a cross section is linear in $y$, as given by Eq. 6.8 and indicated in Fig. 6.10.[3]

[redacted] (6.8)




(a) (b)

**FIGURE 6.10** The flexural stress distribution at a cross section where $\rho(x)$ is positive.

[3] In Section 6.5 we will consider stresses in nonhomogeneous beams, that is, stresses in beams that are made of more than one material.

As indicated in Fig. 6.10, the stress resultants that are related to the normal stress $\sigma_x$ acting on the cross section are:

$$F(x) = \int_A \sigma_x \, dA, \qquad M(x) = -\int_A y\sigma_x \, dA \tag{6.9}$$

A positive moment produces compression in the $+y$ fibers of the beam.

In Section 9.4 we will consider axial deformation combined with bending, but, for the present discussion of bending alone, let $F \equiv 0$. Therefore, substituting Eq. 6.8 into Eqs. 6.9, we get

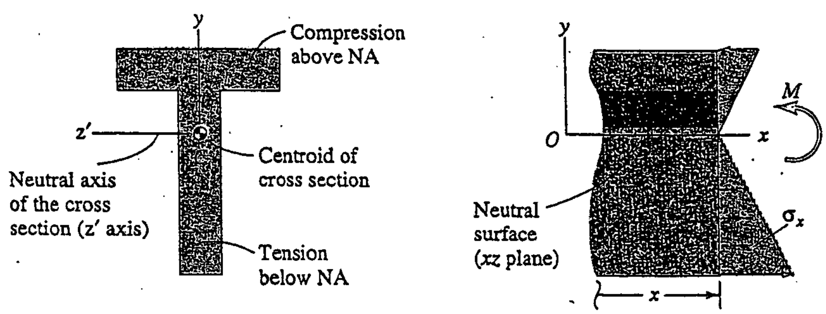
$$F = -\frac{E}{\rho}\int_A y \, dA = 0, \qquad M = \frac{E}{\rho}\int_A y^2 \, dA \tag{6.10}$$

The integrals appearing in Eqs. 6.10 are section properties that are defined in Appendix C:

$$\int_A dA = A, \quad \int_A y \, dA = \bar{y}A, \quad \int_A y^2 \, dA = I_z \tag{6.11}$$

where $A$ is the cross-sectional area, $\bar{y}$ is the $y$ coordinate of the *centroid* of the cross section, and $I_z$ is the *area moment of inertia* about the $z$ axis of the cross section.

In order to satisfy the condition $F = 0$, we must make $\bar{y} = 0$. That is, the $z$ axis of the cross section (labeled the $z'$ axis in Figs. 6.10*a* and 6.11*a*) must pass through the centroid of the cross section. Thus, **the $x$ axis passes through the centroid of each cross section of the undeformed beam.** The $z'$ axis is called the *neutral axis of the cross section,* or simply the *neutral axis* (*NA*), because it is the boundary between the portion of the cross section that is in compression and the portion that is in tension, as indicated in Fig. 6.11.[4]




(*a*) Cross section. (*b*) Profile view.

**FIGURE 6.11** (*a*) The location of the neutral axis of the cross section, and (*b*) the flexural stress distribution for a homogeneous beam in bending.

[4]In the future, the "z axis of the cross section" will just be labeled $z$, not $z'$, even though the true $z$ axis does not lie in the particular cross section under consideration.

Combining Eqs. 6.10b and 6.11c, we obtain the *moment-curvature equation* of Bernoulli-Euler beam theory, namely

[illegible] **Moment-curvature equation** (6.12)

The *curvature* $\kappa(x)$ is related to the *radius of curvature* $\rho(x)$ by $\kappa(x) = \dfrac{1}{\rho(x)}$. The product $EI$ is called the *flexural rigidity* of the beam. (In Eq. 6.12 the subscript has been dropped from $I_z$ to simplify the remainder of the discussion of bending of symmetric beams. Subscripts will be needed again in the Section 6.6 on Unsymmetric Bending.)

We can relate the moment-curvature equation, Eq. 6.12, to the deformed-beam segments in Fig. 6.8 by noting that a positive bending moment, $M(x)$, leads to a positive value of $\rho(x)$, which means that the beam is concave upward, as shown in Fig. 6.8*a*. Conversely, a negative moment produces a negative curvature, which means that the center of curvature lies in the $-y$ direction, as shown in Fig. 6.8*b*.

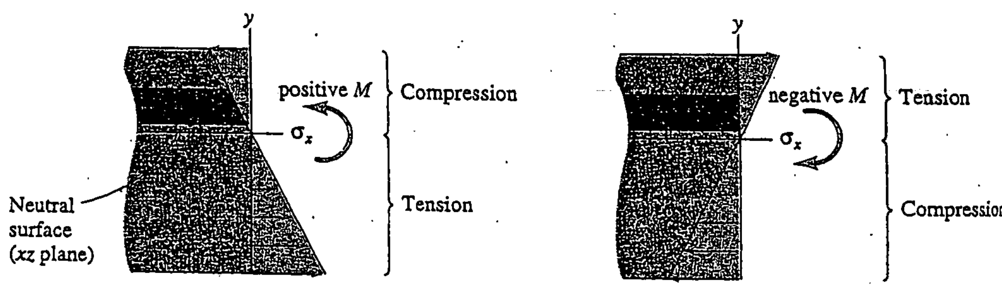
Finally, Eqs. 6.8 and 6.12 may be combined to give the important *flexure formula* of Bernoulli-Euler beam theory.[5]

[illegible] **Flexure formula** (6.13)

By making the assumptions that plane sections remain plane and that the material is linearly elastic with $E = E(x)$, we have obtained an expression for the stress distribution on a cross section subjected to bending moment $M(x)$. This is the linear stress distribution illustrated in Fig. 6.12.[6]

An assumption made in the derivation of the flexure formula, Eq. 6.13, is that $\sigma_x$ is much greater than either $\sigma_y$ or $\sigma_z$. It is left as an exercise for the reader to show that this is a reasonable assumption if the beam is long in comparison with its cross-sectional dimensions. (Homework Problem 6.3-36)

**FIGURE 6.12** The flexural stress distribution in a linearly elastic beam.




(*a*) Positive moment. (*b*) Negative moment.

[5]According to the sign convention adopted in this text and illustrated in Fig. 5.4, a positive moment produces compression in the $+y$ fibers of the beam. This results in a minus sign in Eq. 6.13. Some textbooks adopt a different sign convention that leads to a plus sign in the flexure formula.

[6]Compressive stresses as well as tensile stresses may be shown acting on the cross section, as in Fig. 6.11*b*. However, to emphasize here that $\sigma_x$ is linear in $y$, compressive stresses are shown in Fig. 6.12 as a continuation of the straight-line plot that depicts tensile stresses.